Matroids and Hyperplane Arrangements

Part Two

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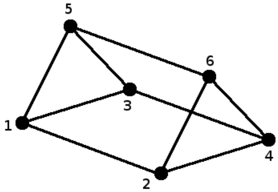

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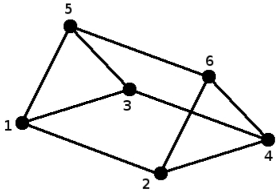

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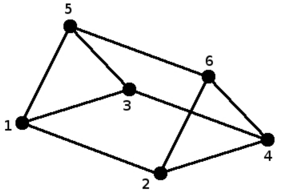

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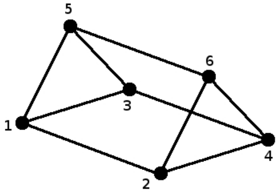

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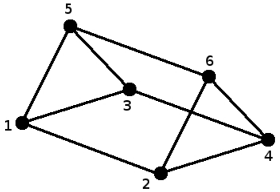

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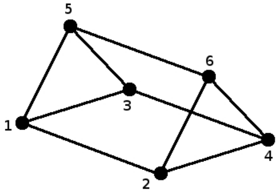
- X is a **flat** if $X = cl(X)$.



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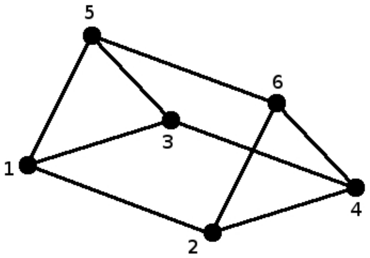

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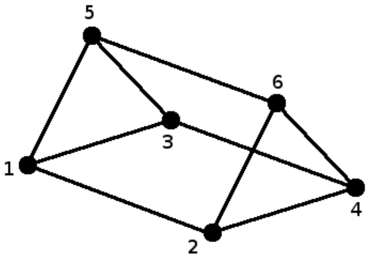
- An arrangement is **2-generic** if for every flat X with $\mathrm{Rank}(X) \leq 2$, $\mathrm{Rank}(X) = |X|$.



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- We do this by finding a matrix Λ, whose rows correspond to the hyperplanes in $\mathcal{A}$, that satisfies certain properties so that the columns of Λ correspond to elements in $\mathcal{M}^2$.

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For the prism example, let $\mathfrak{X} = \{1234, 1256, 3456\}$. Then

$$J_{\mathfrak{X}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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- For the third property, we check that Γ, the set of maximal 2-cliques, satisfies the neighborly condition.

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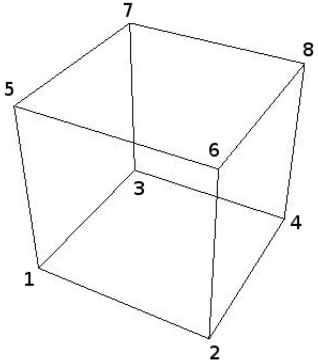
(2') Generalization of condition (2). If $X - \{i\}$ is contained in a 2-clique for some $i \in X$, then so is X.

Consider the hyperplane arrangement

$$\mathcal{A} = \{x \pm y, y \pm z, z \pm w, w \pm x\}.$$

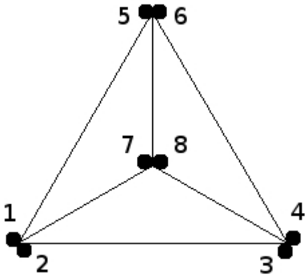
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$



$$\mathfrak{X} = \{1357, 2358, 1458, 2457, 1368, 2367, 1467, 2468\}$$

$$J_{\mathfrak{X}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

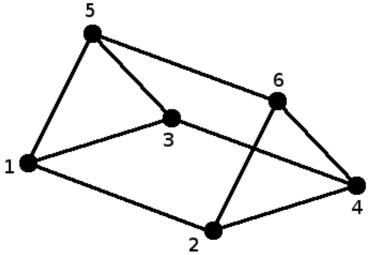
$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

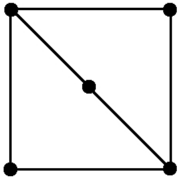


This tells us that in the OS Algebra for this arrangement,

$$\underbrace{[(e_1 + e_2) - (e_7 + e_8)]}_{a}\underbrace{[(e_3 + e_4) - (e_7 + e_8)]}_{b}\underbrace{[(e_5 + e_6) - (e_7 + e_8)]}_{c} = 0$$

Conjecture





Sources

- *Arrangements of Hyperplanes* by Peter Orlik and Hiroaki Terao
- *Matroid Theory* by James Oxley
- *Determining Resonance Varieties of Hyperplane Arrangements* by Andres Perez
- The brain of Dr. Michael Falk.